

# LOWER BOUNDS FOR THE DYADIC HILBERT TRANSFORM

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**ABSTRACT.** In this paper, we seek lower bounds of the dyadic Hilbert transform (Haar shift) of the form  $\|\mathbb{H}f\|_{L^2(K)} \geq C(I, K)\|f\|_{L^2(I)}$  where  $I$  and  $K$  are two dyadic intervals and  $f$  supported in  $I$ . If  $I \subset K$  such bounds exist while in the other cases  $K \subsetneq I$  and  $K \cap I = \emptyset$  such bounds are only available under additional constraints on the derivative of  $f$ . In the later case, we establish a bound of the form  $\|\mathbb{H}f\|_{L^2(K)} \geq C(I, K)|\langle f \rangle_I|$  where  $\langle f \rangle_I$  is the mean of  $f$  over  $I$ . This sheds new light on the similar problem for the usual Hilbert transform.

**Résumé** Dans cet article, nous établissons des bornes pour la transformée de Hilbert dyadique (*Haar shift*) de la forme  $\|\mathbb{H}f\|_{L^2(K)} \geq C(I, K)\|f\|_{L^2(I)}$  où  $I$  et  $K$  sont des intervalles dyadiques et  $f$  est à support dans  $I$ . Si  $I \subset K$  de telles bornes existent sans condition supplémentaire sur  $f$  alors que dans les cas  $K \subsetneq I$  et  $K \cap I = \emptyset$  une telle borne n'existe que si on impose une condition sur la dérivée de  $f$ . Dans le dernier cas nous établissons une borne de la forme  $\|\mathbb{H}f\|_{L^2(K)} \geq C(I, K)|\langle f \rangle_I|$  où  $\langle f \rangle_I$  est la moyenne de  $f$  sur  $I$ . Ce travail permet ainsi une meilleure compréhension du problème similaire pour la transformée de Hilbert sur  $\mathbb{R}$ .

## 1. INTRODUCTION

The aim of this paper is to establish lower bounds on the dyadic Hilbert transform (Haar shift) in the spirit of those that are known for the usual Hilbert transform.

The Hilbert transform is one of the most ubiquitous and important operators in harmonic analysis. It can be defined on  $L^2(\mathbb{R})$  as the Fourier multiplier  $\widehat{Hf}(\xi) = -i \operatorname{sgn}(\xi) \widehat{f}(\xi)$  which shows that  $H : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  is a unitary bijection. Alternatively, the Hilbert transform is defined via

$$Hf(x) = \frac{1}{\pi} \text{p. v.} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy.$$

While boundedness of this operator is by now rather well understood, obtaining lower bounds for the truncated Hilbert transform is still an ongoing task. More precisely, we are looking for bounds of the form  $\|\mathbf{1}_K Hf\|_{L^2(\mathbb{R})} \gtrsim \|f\|_{L^2(\mathbb{R})}$  (for some set  $K \subsetneq \mathbb{R}$  and  $f$  satisfying some additional constraint). Without additional constraints, such an inequality can of course not hold and a first restriction one usually imposes is that  $f$  is supported in some interval  $I$ . Before describing existing literature, let us first motivate the question.

The most well known application of the Hilbert transform comes from complex analysis. Indeed, if  $F$  is a reasonably decaying holomorphic function on the upper half-plane, then its boundary value  $f$  satisfies  $Hf = -if$ . In particular, its real and imaginary parts are connected via  $\operatorname{Im}(f) = H\operatorname{Re}(f)$  and  $\operatorname{Re}(f) = -H\operatorname{Im}(f)$ . Conversely, if  $f$  is a reasonable real valued function, say  $f \in L^2(\mathbb{R})$  with  $\operatorname{supp} f \subset I$ ,  $I$  some interval, then  $\tilde{f} := f + iHf$  is the boundary value of a holomorphic function in the upper half-plane. The question we are asking is whether the knowledge of  $\operatorname{Im}(\tilde{f})$  on some interval

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$K$  determines  $f$  stably. In other words, we are looking for an inequality of the form  $\left\| \operatorname{Im}(\tilde{f}) \right\|_{L^2(K)} \gtrsim \left\| \Re(\tilde{f}) \right\|_{L^2(I)}$ .

An other instance of the Hilbert transform is in the inversion formula of the Radon Transform. Recall from [Na, Chapter II] that the Radon transform of a function  $f \in (\mathbb{R}^2)$  is defined by

$$Rf(\theta, s) = \int_{\langle x, \theta \rangle = s} f(x) \, dx, \quad \theta \in \mathbb{S}^1, s \in \mathbb{R}$$

while the inversion formula reads

$$f(x) = \frac{1}{4\pi} \int_{\mathbb{S}^1} H_s[\partial_s Rf(\theta, \cdot)](\theta, \langle x, \theta \rangle) \, d\sigma(\theta)$$

where the Hilbert transform acts in the  $s$ -variable. In practice,  $Rf(\theta, s)$  can only be measured for  $s$  in a given interval  $K$  which may differ from the relevant interval for  $f$ . This is a second (and main) motivation for establishing lower bounds on the Hilbert transform which should lead to estimates of stable invertibility of the restricted view Radon transform. The introduction of [CNDK] provides nice insight on this issue.

It turns out that the relative position of the intervals  $I$  and  $K$  plays a central role here and we distinguish four cases:

- *Covering.* When  $K \supsetneq I$  the inversion is stable and an explicit inversion formula is known [Tr].
- *Interior problem.* When  $K \subset I$ , stable reconstruction is no longer possible. This case, known as the *interior problem* in tomography has been extensively studied (see e.g. [CNDK, Ka2, KKW, KCND, YYW]).
- *Gap.* When  $I \cap K = \emptyset$ , the singular value decomposition of the underlying operator has been given in [Ka1] and this case was further studied by Alaifari, Pierce, and Steinerberger in [APS]. It turns out that oscillations of  $f$  imply instabilities of the problem. The main result of [APS] is that there exists constants  $c_1, c_2$  depending only on  $I, K$  such that, for every  $f \in H^1(I)$ ,

$$\|Hf\|_{L^2(K)} \geq c_1 \exp\left(-c_1 \frac{\|f'\|_{L^2(I)}}{\|f\|_{L^2(I)}}\right) \|f\|_{L^2(I)}.$$

Moreover, the authors conjecture that  $\|f'\|_{L^2(I)}$  may be replaced by  $\|f'\|_{L^1(I)}$ .

- *Overlap.* When  $I \cap K \neq \emptyset$  and  $I \cap (\mathbb{R} \setminus K) \neq \emptyset$ , a pointwise stability estimate has been shown in [DNCK] while the spectral properties of the underlying operator are the subject of [AK, ADK].

Most proofs go through spectral theory. More precisely, the strategy of proof is the same as for the similar problem for the Fourier transform. Recall that in their seminal work on time-band limiting, Landau, Pollak, Slepian found a differential operator that commutes with the “time-band” limiting operator (see [Sl] for an overview of the theory and further references). The spectral properties of this differential operator are relatively easy to study and the spectral properties of the “time-band” limiting operator then follow. The counter-part of this strategy is that it relies on a “happy accident” (as termed by Slepian) that does not shed light on the geometric/analytic features at play in the Hilbert transform. Therefore, no hint towards lower bounds for more general Calderon-Zygmund operators, nor towards the conjecture in [APS] is obtained through that approach.

Our aim here is precisely to shed new light on lower bounds for the truncated Hilbert transform. To do so, we follow the current paradigm in harmonic analysis by replacing the Hilbert transform by its dyadic version (Haar shift) which serves at first as a toy model. We then study the *gap*, *covering* and *interior* problems for the Haar shift.

To be more precise, let  $\mathcal{D}$  be the set of dyadic intervals. To a dyadic interval  $I$ , we associate the Haar function  $h_I = |I|^{-1/2}(\mathbf{1}_{I_+} - \mathbf{1}_{I_-})$  where  $I_{\pm}$  are the sons of  $I$  and  $|I|$  its the length. The dyadic Hilbert transform (Haar shift) is defined by

$$\mathbb{H}f = \sum_{I \in \mathcal{D}} \langle f, h_I \rangle \mathbb{H}h_I$$

where  $\mathbb{H}h_I = 2^{-1/2}(h_{I_+} - h_{I_-})$  (see the beginning of the next section for more details). One can define a similar transform for generalized dyadic intervals obtained by dilating and properly translating  $\mathcal{D}$ . It turns out that the usual Hilbert transform is the average over a suitable family of generalized dyadic intervals of the corresponding Haar shifts, see [Pet2, Pet1, Hy]. This approach has been very successful for upper bounds but it seems much less adapted to lower bounds; though we point to two cases in [NV, NRVV] where lower estimates for the martingale transforms are obtained and provide related lower estimates for the Hilbert transform.

Nevertheless, the Haar shift shares many common features with the continuous Hilbert transform, and this is why we here establish lower bounds for this transform. We hope those lower bounds give some insight on the problem of establishing lower bounds for the truncated Hilbert transform. However, our results depend heavily on the particular structure of the Haar shift we consider. It would be interesting and probably challenging to extend our computations to general shifts and in particular to Haar multipliers of fixed sign pattern. Since we are dealing with a very particular Haar shift we are able to obtain precise formulas and estimates by direct computations, see e.g. equation (3.3) below. It would be interesting to establish similar formulas for general dyadic shifts as defined in [Hy2].

The main result we obtain is the following:

**Theorem.** *Let  $I, K$  be two dyadic intervals. Then*

- (1) *Covering. If  $I \subset K$  then  $\|\mathbf{1}_K \mathbb{H}f\|_2 \geq \frac{1}{2}\|f\|_2$  for every  $f \in L^2(\mathbb{R})$  with  $\text{supp } f \subset I$ .*
- (2) *Gap. If  $I \cap K = \emptyset$ , then no estimate of the form  $\|\mathbf{1}_K \mathbb{H}f\|_2 \gtrsim \|f\|_2$  holds for every  $f \in L^2(\mathbb{R})$  with  $\text{supp } f \subset I$ . But*
  - either  $I \subset [2^{M-1}, 2^M]$  and  $K \subset [0, 2^{M-2}]$  for some integer  $M$ , then  $\mathbf{1}_K \mathbb{H}f = 0$  for every  $f \in L^2(\mathbb{R})$  with  $\text{supp } f \subset I$*
  - or for every  $0 < \eta < 1$ , there exists  $C = C(I, K, \eta)$  such that  $\|\mathbf{1}_K \mathbb{H}f\|_2 \geq C\|f\|_2$  for every  $f \in L^2(\mathbb{R})$  of the form  $f = f_0 \mathbf{1}_I$  with  $f_0 \in W^{1,2}(\mathbb{R})$  and  $|I| \|f'_0\|_{L^2(I)} \leq 2\pi\eta \|f_0\|_{L^2(I)}$ .*
- (3) *Interior problem. If  $K \subset I$ , then no estimate of the form  $\|\mathbf{1}_K \mathbb{H}f\|_2 \gtrsim \|f\|_2$  holds for every  $f \in L^2(\mathbb{R})$  with  $\text{supp } f \subset I$ . But  $\|\mathbf{1}_K \mathbb{H}f\|_2 \geq \|\mathbf{1}_K f\|_2$  for every  $f \in L^2(\mathbb{R})$  with  $\text{supp } f \subset I$ .*

Note that the fact that we assume that both  $I, K$  are dyadic implies that the overlapping case does not occur here. In the Gap case, we actually show that  $\mathbf{1}_K \mathbb{H}f = C(I, K) \int_I f(x) dx$ . Therefore, if  $f$  has zero mean, then its Haar shift is zero outside its support. This is a major difference with the Hilbert transform which only has extra decay in that case. As a consequence, one can not recover functions with zero-mean from their Haar shift outside the support. To avoid this situation, one may

use the Poincaré-Wirtinger inequality to control the mean of  $f$  by its  $L^2$ -norm when  $f$  has small derivative.

In Section 2 we collect basic facts and notation and Sections 3, 4, and 5 are then devoted each to one of the cases that arise in our main theorem.

## 2. NOTATIONS AND COMPUTATIONS OF INTEREST

In this paper, all functions will be in  $L^2(\mathbb{R})$ . We write

$$\|f\|_{L^2} = \left( \int_{\mathbb{R}} |f(x)|^2 dx \right)^{1/2}, \quad \langle f, g \rangle_{L^2} = \int_{\mathbb{R}} f(x) \overline{g(x)} dx.$$

For  $I$  an interval of finite length  $|I|$  and  $f \in L^2(\mathbb{R})$ , we write

$$\langle f \rangle_I = \frac{1}{|I|} \int_I f(x) dx$$

for the mean of  $f$  over  $I$ .

Let  $\mathcal{D}$  denote the collection of dyadic intervals on  $\mathbb{R}$ , namely the intervals of the form  $\mathcal{D} = \{[2^k \ell, 2^k(\ell+1)) : k, \ell \in \mathbb{Z}\}$ . For  $I = [2^k \ell, 2^k(\ell+1))$ , we denote the children of  $I$  by  $I_- = [2^k \ell, 2^k(\ell+1/2)) = [2^{k-1} 2\ell, 2^{k-1}(2\ell+1)) \in \mathcal{D}$  and  $I_+ = [2^k(\ell+1/2), 2^k(\ell+1)) = [2^{k-1}(2\ell+1), 2^{k-1}(2\ell+2)) \in \mathcal{D}$ . The parent of  $I$ , denoted  $\widehat{I}$ , is the unique interval in  $\mathcal{D}$  such that  $I = \widehat{I}_{\varepsilon(I)}$  with  $\varepsilon(I) \in \{\pm 1\}$ .

We will frequently use the following computations: If  $\mathcal{L} \in \mathcal{D}$ , then

$$(2.1) \quad \sum_{L \in \mathcal{D}, L \supsetneq \mathcal{L}} \frac{1}{|L|} = \frac{1}{|\mathcal{L}|} \sum_{k=1}^{\infty} 2^{-k} = \frac{1}{|\mathcal{L}|}$$

while for  $\mathcal{L} \subsetneq \mathcal{K} \in \mathcal{D}$

$$(2.2) \quad \sum_{L \in \mathcal{D}, \mathcal{L} \subsetneq L \subsetneq \mathcal{K}} \frac{1}{|L|} = \frac{1}{|\mathcal{L}|} \left( 1 - \frac{|\mathcal{L}|}{|\mathcal{K}|} \right).$$

These results follow from the fact that for every  $k \geq 1$  there is a unique  $L \supsetneq \mathcal{L}$  with  $|L| = 2^k |\mathcal{L}|$ .

For  $I \in \mathcal{D}$ , we denote by  $h_I$  the corresponding Haar function,

$$h_I = \frac{-\mathbf{1}_{I_-} + \mathbf{1}_{I_+}}{\sqrt{|I|}}.$$

Note that, if  $K \in \mathcal{D}$  is such that  $K \subset I_{\pm}$  then  $h_I$  is constant on  $K$ . Then, denoting by  $c(K)$  the center of  $K$ ,  $h_I(K) = h_I(c(K)) = \frac{\varepsilon(I, K)}{\sqrt{|I|}}$  where  $\varepsilon(I, K) \in \{\pm 1\}$ . Also,  $h_I$  has mean zero so that  $\langle \mathbf{1}_I, h_I \rangle_{L^2} = 0$  and, more generally, if  $I \subset J$ ,  $\langle \mathbf{1}_J, h_I \rangle_{L^2} = 0$ .

Recall that  $\{h_I : I \in \mathcal{D}\}$  is an orthonormal basis of  $L^2(\mathbb{R})$ . In particular, if  $f \in L^2(\mathbb{R})$  and  $I \in \mathcal{D}$ , we write  $\widehat{f}(I) = \langle f, h_I \rangle_{L^2}$  so that

$$f = \sum_{I \in \mathcal{D}} \widehat{f}(I) h_I$$

and, for  $f, g \in L^2(\mathbb{R})$ ,

$$\langle f, g \rangle_{L^2} = \sum_{I \in \mathcal{D}} \widehat{f}(I) \overline{\widehat{g}(I)}.$$

Further, when  $f \in L^2(\mathbb{R})$  is supported on an interval  $I \in \mathcal{D}$ , then it is simpler to write

$$(2.3) \quad f = \langle f \rangle_I \mathbf{1}_I + \sum_{J \subset I} \hat{f}(J) h_J$$

from which it follows that

$$(2.4) \quad \|f\|_{L^2}^2 = \langle f \rangle_I^2 |I| + \sum_{J \subset I} |\hat{f}(J)|^2$$

since  $\mathbf{1}_I$  and  $h_J$  are orthogonal when  $J \subset I$ . On the other hand

$$(2.5) \quad \mathbf{1}_I = \sum_{L \in \mathcal{D}} \langle \mathbf{1}_I, h_L \rangle_{L^2} h_L = \sum_{L \supseteq I} \langle \mathbf{1}_I, h_L \rangle_{L^2} h_L = |I| \sum_{L \supseteq I} h_L(I) h_L$$

since  $\langle \mathbf{1}_I, h_L \rangle_{L^2} = \int_I h_L(x) dx = 0$  when  $L \subset I$ .

Let  $\mathbb{H}$  denote the *dyadic Hilbert transform* (the Haar shift) which is the bounded linear operator on  $L^2(\mathbb{R})$  defined by

$$\mathbb{H}h_I = \frac{h_{I_+} - h_{I_-}}{\sqrt{2}}.$$

Note that  $\mathbb{H}h_I$  is supported on  $I$ . It is easily seen that  $\langle \mathbb{H}h_I, \mathbb{H}h_J \rangle_{L^2} = \delta_{I,J}$  so that  $\mathbb{H}$  is a *unitary* transform.

We will now make a few simple observations.

(1) If  $K$  is any dyadic interval then the function  $\mathbf{1}_K \mathbb{H}h_L$  is supported on  $K \cap L$ . In particular, if  $L \subset K$ ,  $\mathbf{1}_K \mathbb{H}h_L = \mathbb{H}h_L$ .

(2) If  $L \supsetneq \hat{K}$ , then the function  $\mathbf{1}_K \mathbb{H}h_L = \frac{\varepsilon(K, L)}{\sqrt{|L|}} \mathbf{1}_K$  where  $\varepsilon(K, L) \in \{\pm 1\}$ . We will write

$$\mathbf{1}_K \mathbb{H}h_L = \mathbb{H}h_L(K) \mathbf{1}_K \text{ where again } \mathbb{H}h_L(K) = \mathbb{H}h_L(c(K)).$$

Indeed,  $K = \hat{K}_{\varepsilon(K)} \subsetneq L$  thus  $\hat{K}_{\varepsilon(K)} \subset L_{\pm}$  but then

$$\mathbf{1}_K \mathbb{H}h_L = \pm \mathbf{1}_K \frac{h_{L_{\pm}}}{\sqrt{2}} = \pm \frac{h_{L_{\pm}}(K)}{\sqrt{2}} \mathbf{1}_K$$

which is of the desired form.

(3) If  $L = \hat{K}$ , then  $K = L_{\varepsilon(K)}$  and  $\mathbf{1}_K \mathbb{H}h_L = \frac{\varepsilon(K) h_K}{\sqrt{2}}$ .

When  $f \in L^2(\mathbb{R})$  is supported in  $I \in \mathcal{D}$ , from the decomposition (2.3), we obtain

$$(2.6) \quad \mathbf{1}_K \mathbb{H}f = \langle f \rangle_I \mathbf{1}_K \mathbb{H} \mathbf{1}_I + \sum_{J \subset I} \hat{f}(J) \mathbf{1}_K \mathbb{H}h_J.$$

On the other hand, from the decomposition (2.5), we have that for any  $I, K \in \mathcal{D}$ :

$$\mathbb{H} \mathbf{1}_I = |I| \sum_{L \supseteq I} h_L(I) \mathbb{H}h_L$$

thus

$$(2.7) \quad \mathbf{1}_K \mathbb{H} \mathbf{1}_I = |I| \sum_{L \supseteq I} h_L(I) \mathbf{1}_K \mathbb{H}h_L.$$

We can now prove the following

**Lemma 2.1.** *For  $I \in \mathcal{D}$ ,  $\mathbf{1}_I \mathbb{I} \mathbf{1}_I = \sqrt{|I|} h_I$ .*

*Proof.* Let  $K = I_{\pm}$ . We want to prove that

$$\mathbf{1}_{I_{\pm}} \mathbb{I} \mathbf{1}_I = \pm \mathbf{1}_{I_{\pm}}.$$

From (2.7), we deduce that

$$\mathbf{1}_K \mathbb{I} \mathbf{1}_I = |I| \left[ \sum_{L \supsetneq I} h_L(I) \mathbb{I} h_L(K) \right] \mathbf{1}_K.$$

since  $L \supsetneq \widehat{K}$  for any  $I \subsetneq L$ . Observe that the sign of  $h_L(I) \mathbb{I} h_L(K)$ ,  $I \subsetneq L$ , only depends on the position of  $K$  regarding  $I_-$  or  $I_+$ . Indeed, if we have  $K = I_-$  and  $I \subset L_-$  then  $h_L(I) \mathbb{I} h_L(K) = \frac{-1}{|L|}$  with  $h_L(I) = \frac{-1}{\sqrt{|L|}} = -\mathbb{I} h_L(K)$  since  $K = I_- \subset (L_-)_-$ . On the other hand, if  $K = I_-$  and  $I \subset L_+$  then  $h_L(I) \mathbb{I} h_L(K) = \frac{-1}{|L|}$  with  $h_L(I) = \frac{1}{\sqrt{|L|}} = -\mathbb{I} h_L(K)$  since  $K \subset I_- \subset (L_+)_-$ . Similar arguments lead to  $h_L(I) \mathbb{I} h_L(K) = \frac{1}{|L|}$  when  $K = I_+$  and  $I \subset L_-$  and when  $K = I_+$  and  $I \subset L_+$ . Thus, we obtain

$$\begin{aligned} \mathbf{1}_K \mathbb{I} \mathbf{1}_I &= \varepsilon(K, I) |I| \left[ \sum_{L \supsetneq I} \frac{1}{|L|} \right] \mathbf{1}_K \\ &= \varepsilon(K, I) |I| \left[ \sum_{k=1}^{\infty} \frac{1}{2^k |I|} \right] \mathbf{1}_K \\ &= \varepsilon(K, I) \mathbf{1}_K \end{aligned}$$

as announced.  $\square$

Our aim is to obtain lower bounds of  $\|\mathbf{1}_K \mathbb{I} f\|_2$  when  $f \in L^2(\mathbb{R})$  is supported in  $I \in \mathcal{D}$ . This requires an understanding of  $\mathbf{1}_K \mathbb{I} \mathbf{1}_I$  in the three cases  $K \subset I$ ,  $I \subset K$  and  $K \cap I = \emptyset$ .

### 3. FIRST CASE: $I \subset K$

This is the “easy” and most favorable case:

**Theorem 3.1.** *Let  $I \subset K \in \mathcal{D}$ . Then, for every  $f \in L^2(\mathbb{R})$  supported in  $I$ ,*

$$\|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 \geq \left(1 - \frac{3}{4} \frac{|I|}{|K|}\right) \|f\|_{L^2}^2.$$

*Proof.* According to (2.7) we have

$$\begin{aligned} (3.1) \quad \mathbf{1}_K \mathbb{I} f &= \langle f \rangle_I \mathbf{1}_K \mathbb{I} \mathbf{1}_I + \sum_{J \subset I} \widehat{f}(J) \mathbf{1}_K \mathbb{I} h_J \\ &= \langle f \rangle_I \mathbf{1}_K \mathbb{I} \mathbf{1}_I + \sum_{J \subset I} \widehat{f}(J) \mathbb{I} h_J. \end{aligned}$$

Indeed, notice that in (3.1),  $J \subset I \subset K$  so that  $\mathbb{I} h_J$  is supported in  $J \subset K$  and  $\mathbf{1}_K \mathbb{I} h_J = \mathbb{I} h_J$ .

Now we further have that:

$$\begin{aligned}
 \|\mathbf{1}_K \mathbb{H} f\|_{L^2}^2 &= \langle \mathbf{1}_K \mathbb{H} f, \mathbf{1}_K \mathbb{H} f \rangle_{L^2} \\
 &= \left\langle \langle f \rangle_I \mathbf{1}_K \mathbb{H} \mathbf{1}_I + \sum_{J \subset I} \hat{f}(J) \mathbb{H} h_J, \langle f \rangle_I \mathbf{1}_K \mathbb{H} \mathbf{1}_I + \sum_{J \subset I} \hat{f}(J) \mathbb{H} h_J \right\rangle_{L^2} \\
 &= \langle f \rangle_I^2 \|\mathbf{1}_K \mathbb{H} \mathbf{1}_I\|_{L^2}^2 + \left\langle \mathbb{H} \left( \sum_{J \subset I} \hat{f}(J) h_J \right), \mathbb{H} \left( \sum_{J \subset I} \hat{f}(J) h_J \right) \right\rangle_{L^2} \\
 (3.2) \quad &+ 2 \langle f \rangle_I \sum_{J \subset I} \langle \mathbb{H} \mathbf{1}_I, \mathbb{H} h_J \rangle_{L^2} \hat{f}(J).
 \end{aligned}$$

But, as  $\mathbb{H}$  is unitary,  $\langle \mathbb{H} \mathbf{1}_I, \mathbb{H} h_J \rangle_{L^2} = \langle \mathbf{1}_I, h_J \rangle_{L^2} = 0$  since  $J \subset I$ . Further, using again that  $\mathbb{H}$  is unitary and that the  $\{h_J\}$ 's are orthonormal,

$$\begin{aligned}
 \left\langle \mathbb{H} \left( \sum_{J \subset I} \hat{f}(J) h_J \right), \mathbb{H} \left( \sum_{J \subset I} \hat{f}(J) h_J \right) \right\rangle_{L^2} &= \left\langle \sum_{J \subset I} \hat{f}(J) h_J, \sum_{J \subset I} \hat{f}(J) h_J \right\rangle_{L^2} \\
 &= \sum_{J \subset I} |\hat{f}(J)|^2.
 \end{aligned}$$

Therefore (3.2) reduces to

$$\|\mathbf{1}_K \mathbb{H} f\|_{L^2}^2 = \langle f \rangle_I^2 \|\mathbf{1}_K \mathbb{H} \mathbf{1}_I\|_{L^2}^2 + \sum_{J \subset I} |\hat{f}(J)|^2.$$

As  $\|\mathbf{1}_K \mathbb{H} \mathbf{1}_I\|_{L^2}^2 \leq \|\mathbb{H} \mathbf{1}_I\|_{L^2}^2 = |I|$ , we get

$$(3.3) \quad \|\mathbf{1}_K \mathbb{H} f\|_{L^2}^2 \geq \frac{\|\mathbf{1}_K \mathbb{H} \mathbf{1}_I\|_{L^2}^2}{|I|} \left( \langle f \rangle_I^2 |I| + \sum_{J \subset I} |\hat{f}(J)|^2 \right) = \frac{\|\mathbf{1}_K \mathbb{H} \mathbf{1}_I\|_{L^2}^2}{|I|} \|f\|_{L^2}^2.$$

It remains to estimate  $\|\mathbf{1}_K \mathbb{H} \mathbf{1}_I\|_{L^2}^2$  from below. Recall from (2.7) that

$$\begin{aligned}
 \frac{1}{|I|} \mathbf{1}_K \mathbb{H} \mathbf{1}_I &= \sum_{L \supsetneq I} h_L(I) \mathbf{1}_K \mathbb{H} h_L = \left( \sum_{L \supsetneq \hat{K}} + \sum_{L = \hat{K}} + \sum_{K \supset L \supsetneq I} \right) h_L(I) \mathbf{1}_K \mathbb{H} h_L \\
 (3.4) \quad &= \left( \sum_{L \supsetneq \hat{K}} h_L(I) \mathbb{H} h_L(K) \right) \mathbf{1}_K + \frac{\varepsilon(K) h_{\hat{K}}(I)}{\sqrt{2}} h_K + \sum_{K \supset L \supsetneq I} h_L(I) \mathbb{H} h_L
 \end{aligned}$$

with the three observations made on  $\mathbf{1}_K \mathbb{H} h_L$ . Now notice that the three terms in (3.4) are orthogonal. Indeed, if  $L \subset K$  then  $h_K$  and  $\mathbb{H} h_L$  are supported in  $K$  and have mean 0. Therefore, they are orthogonal to  $\mathbf{1}_K$ . Further,  $\sqrt{2} \mathbb{H} h_L = h_{L_+} - h_{L_-}$  and  $L_{\pm} \subsetneq K$  thus  $h_{L_{\pm}}$  is orthogonal to  $h_K$ . Moreover,

$$\left| \frac{\varepsilon(K) h_{\hat{K}}(I)}{\sqrt{2}} \right| = \frac{1}{\sqrt{2|\hat{K}|}} = \frac{1}{2\sqrt{|K|}}$$

and, as  $\mathbb{III}$  is unitary, the  $\mathbb{III}h_L$ 's are orthonormal. Therefore

$$\begin{aligned} \frac{\|\mathbf{1}_K \mathbb{III} \mathbf{1}_I\|_{L^2}^2}{|I|} &= |I||K| \left( \sum_{L \supsetneq \hat{K}} h_L(I) \mathbb{III} h_L(K) \right)^2 + \frac{|I|}{4|K|} + |I| \sum_{K \supset L \supsetneq I} |h_L(I)|^2 \\ &\geq \frac{|I|}{4|K|} + |I| \sum_{K \supset L \supsetneq I} \frac{1}{|L|}. \end{aligned}$$

Now this last quantity is  $\frac{1}{4} = 1 - \frac{3}{4} \frac{|I|}{|K|}$  when  $K = I$  and (2.2) shows that it is  $1 - \frac{3}{4} \frac{|I|}{|K|}$  when  $K \supsetneq I$ , which completes the proof.  $\square$

#### 4. SECOND CASE: $I \cap K = \emptyset$

Suppose that  $K, I \in \mathcal{D}$  are such that  $K \cap I = \emptyset$ . First observe that

$$\begin{aligned} \mathbf{1}_K \mathbb{III} f &= \langle f \rangle_I \mathbf{1}_K \mathbb{III} \mathbf{1}_I + \sum_{J \subset I} \hat{f}(J) \mathbf{1}_K \mathbb{III} h_J \\ &= \langle f \rangle_I \mathbf{1}_K \mathbb{III} \mathbf{1}_I \end{aligned}$$

with the last equality following since  $\mathbb{III}h_J$  is supported on  $J \subset I$  and that  $I \cap K = \emptyset$  and so  $J \cap K = \emptyset$  as well. Thus, we have that

$$\|\mathbf{1}_K \mathbb{III} f\|_{L^2}^2 = \frac{\|\mathbf{1}_K \mathbb{III} \mathbf{1}_I\|_{L^2}^2}{|I|} \langle f \rangle_I^2 |I|.$$

*Remark.* From this, it is obvious that a lower bound of the form  $\|\mathbf{1}_K \mathbb{III} f\|_{L^2}^2 \geq C \|f\|_{L^2}^2 = \langle f \rangle_I^2 |I| + \sum_{J \subset I} |\hat{f}(J)|^2$  cannot hold without further assumptions on  $f$ . For instance, if  $f$  has mean 0 then  $\mathbf{1}_K \mathbb{III} f = 0$ . One may also restrict attention to non-negative functions in which case the mean would not be zero. However,  $\sum_{J \subset I} |\hat{f}(J)|^2$  may still be arbitrarily large compared to  $\langle f \rangle_I^2 |I|$  so that we would still not obtain a bound of the form  $\|\mathbf{1}_K \mathbb{III} f\|_{L^2}^2 \geq C \|f\|_{L^2}^2$ .

One way to overcome this is to ask for a restriction on the oscillations of  $f$ . For example, when  $f$  is in the Sobolev space  $W^{1,2}(I)$  and  $f'$  its derivative. We extend both  $f$  and  $f'$  by 0 outside the interval  $I$  (so that  $f'$  needs not be the distributional derivative of  $f$  over  $\mathbb{R}$ ). Alternatively,  $f'$  may be defined as the derivative of the Fourier series of  $f$  and extended by 0 outside  $I$ , see below. By Poincaré-Wirtinger (see e.g. [Da, Chap 4] or [ABM, Chap 5]) we have that:

$$(4.1) \quad \|f - \langle f \rangle_I \mathbf{1}_I\|_{L^2(I)} \leq \frac{|I|}{2\pi} \|f'\|_{L^2(I)}.$$

Now, suppose that the norm of the derivative is controlled relative to the norm of the function:

$$(4.2) \quad \|f'\|_{L^2(I)} \leq \eta \frac{2\pi \|f\|_{L^2(I)}}{|I|}, \quad 0 \leq \eta < 1,$$

then we will have that:

$$\begin{aligned} \|f\|_{L^2(I)} &\leq \|f - \langle f \rangle_I \mathbf{1}_I\|_{L^2(I)} + |I|^{\frac{1}{2}} |\langle f \rangle_I| \\ &\leq \eta \|f\|_{L^2(I)} + |I|^{\frac{1}{2}} |\langle f \rangle_I|, \end{aligned}$$



which upon rearrangement will give

$$|I| \langle f \rangle_I^2 \geq (1 - \eta)^2 \|f\|_{L^2(I)}^2.$$

In other words, function satisfying (4.1) are small zero-mean perturbations of constants. For instance, with  $I = [0, 1]$ , let  $(a_k)_{k \in \mathbb{Z} \setminus \{0\}}$  be a sequence such that  $\alpha^2 := \sum_{k \neq 0} k^2 |a_k|^2 < +\infty$ , and  $a_0 = \frac{\alpha}{\eta}$ . We may then define  $f(t) = \sum_{k \in \mathbb{Z}} a_k e^{2ik\pi t}$  on  $[0, 1]$  where the series converges uniformly and extend  $f$  by 0 outside  $[0, 1]$ . On  $[0, 1]$  the weak derivative of  $f$  is given by  $f'(t) = \sum_{k \in \mathbb{Z}} 2ik\pi a_k e^{2ik\pi t}$  where the sum of the series is taken in the  $L^2([0, 1])$  sense (and needs not be extended outside  $[0, 1]$ ). It follows that  $f$  satisfies (4.2).

One can replace the Poincaré-Wirtinger inequality by versions where one tests the  $L^p$  norm of the derivative and the  $L^2$  norm of the function. For such inequalities, we refer to [ABM, Chap 5].

We now turn to computing a lower bound of  $\frac{\|\mathbf{1}_K \mathbb{I} \mathbf{1}_I\|_{L^2}^2}{|I|}$ . First,  $\mathbb{I} \mathbf{1}_I$  is supported in  $I$  so that  $\mathbf{1}_K \mathbb{I} \mathbf{1}_I = 0$  if  $K \subset \mathbb{R}^\pm$  and  $I \subset \mathbb{R}^\mp$ . We will therefore assume that  $I, K \subset \mathbb{R}^+$ , the case  $I, K \subset \mathbb{R}^-$  then follows from the fact that  $\mathbb{I}$  is “odd”, thus  $\mathbf{1}_K \mathbb{I} \mathbf{1}_I = -\mathbf{1}_{-K} \mathbb{I} \mathbf{1}_{-I}$ .

Let  $K \wedge I$  denote the minimal dyadic interval that contains both  $K$  and  $I$ . Note that  $I, K \neq K \wedge I$ , so that  $I$  and  $K$  belong to different dyadic children of  $K \wedge I$ ; for example if  $I \subset (K \wedge I)_+$  then  $K \subset (K \wedge I)_-$  and a similar statement holds when replacing the appropriate  $+$  and  $-$ . Let us now split the identity (2.7) into three parts

$$\begin{aligned} \frac{\mathbf{1}_K \mathbb{I} \mathbf{1}_I}{|I|} &= \sum_{L \supseteq I} h_L(I) \mathbf{1}_K \mathbb{I} h_L \\ &= \left( \sum_{L \supseteq K \wedge I} + \sum_{L=K \wedge I} + \sum_{K \wedge I \supsetneq L \supseteq I} \right) h_L(I) \mathbf{1}_K \mathbb{I} h_L \\ (4.3) \quad &= \left[ \sum_{L \supseteq K \wedge I} h_L(I) \mathbb{I} h_L(K) \right] \mathbf{1}_K + h_{K \wedge I}(I) \mathbf{1}_K \mathbb{I} h_{K \wedge I} + \sum_{K \wedge I \supsetneq L \supseteq I} h_L(I) \mathbf{1}_K \mathbb{I} h_L \end{aligned}$$

since we have that  $\mathbf{1}_K \mathbb{I} h_L$  takes a constant value as described above when  $L \supsetneq K \wedge I$  and evaluating the sums over the regions in question.

Let us now notice that  $L \cap K = \emptyset$  when  $I \subsetneq L \subsetneq K \wedge I$ . Indeed, suppose this were not the case. It is not possible that  $L \subset K$  since  $I \subset L \subset K$ , which contradicts that  $I \cap K = \emptyset$ . Thus we have that  $I, K \subset L$  and hence  $K \wedge I \subset L$ , contradicting that  $L \subsetneq K \wedge I$ , and so  $L \cap K = \emptyset$  as claimed. It follows that the third term in (4.3) vanishes so that

$$\begin{aligned} \frac{\mathbf{1}_K \mathbb{I} \mathbf{1}_I}{|I|} &= \left[ \sum_{L \supseteq K \wedge I} h_L(I) \mathbb{I} h_L(K) \right] \mathbf{1}_K + h_{K \wedge I}(I) \mathbf{1}_K \mathbb{I} h_{K \wedge I} \\ &= \begin{cases} \left[ \sum_{L \supseteq K \wedge I} h_L(I) \mathbb{I} h_L(K) \right] \mathbf{1}_K + \varepsilon(K) \frac{h_{K \wedge I}(I) h_K}{\sqrt{2}} & \text{if } K \wedge I = \widehat{K} \\ \left[ \sum_{L \supseteq K \wedge I} h_L(I) \mathbb{I} h_L(K) \right] \mathbf{1}_K & \text{if } K \wedge I \supsetneq \widehat{K} \end{cases} \end{aligned}$$

which follows from the properties of  $\mathbf{1}_K \text{III} h_L$  given above.

Thus, we have that:

$$\frac{\|\mathbf{1}_K \text{III} \mathbf{1}_I\|_{L^2}^2}{|I|} = \begin{cases} |I| |K| \left| \sum_{L \supsetneq K \wedge I} h_L(I) \text{III} h_L(K) \right|^2 + \frac{|I|}{2|K \wedge I|} & \text{if } K \wedge I = \widehat{K} \\ |I| |K| \left| \sum_{L \supset K \wedge I} h_L(I) \text{III} h_L(K) \right|^2 & \text{if } K \wedge I \supsetneq \widehat{K}. \end{cases}$$

*Remark.* At this stage, we can observe that, when  $K \wedge I \subsetneq \widehat{K}$ ,

$$\frac{\|\mathbf{1}_K \text{III} \mathbf{1}_I\|_{L^2}^2}{|I|} \leq \frac{1}{4}.$$

Indeed, we have that

$$\begin{aligned} \frac{\|\mathbf{1}_K \text{III} \mathbf{1}_I\|_{L^2}^2}{|I|} &\leq |I| |K| \left( \sum_{L \supsetneq K \wedge I} |h_L(I)| |\text{III} h_L(K)| \right)^2 \\ &= |I| |K| \left( \sum_{L \supsetneq K \wedge I} \frac{1}{\sqrt{|L|}} \frac{1}{\sqrt{|L|}} \right)^2 \\ &= |I| |K| \left( \sum_{L \supsetneq K \wedge I} \frac{1}{|L|} \right)^2 \\ &= \frac{|I| |K|}{|K \wedge I|^2} \leq \frac{1}{4}. \end{aligned}$$

Here the last inequality follows since  $I, K \subsetneq K \wedge I$ , so  $|I|, |K| \leq \frac{1}{2} |K \wedge I|$ .

If  $K \wedge I = \widehat{K}$ , there is an extra term and we get  $\frac{|I|}{2|K \wedge I|} \leq \frac{1}{4}$  from which we deduce that

$$\frac{\|\mathbf{1}_K \text{III} \mathbf{1}_I\|_{L^2}^2}{|I|} \leq \frac{1}{2}.$$

Note that, if  $K \wedge I = \widehat{K}$ , then we write  $K = K_- \cup K_+$  so that  $K_{\pm} \wedge I = \widehat{K}$  and  $\mathbf{1}_K \text{III} \mathbf{1}_I = \mathbf{1}_{K_-} \text{III} \mathbf{1}_I + \mathbf{1}_{K_+} \text{III} \mathbf{1}_I$  is an orthogonal decomposition.

To give an estimation of  $|\sum_{L \supset K \wedge I} h_L(I) \text{III} h_L(K)|^2$  when  $K \wedge I \supsetneq \widehat{K}$ , we use the following lemma.

**Lemma 4.1.** *Let  $\mathcal{L}^{(0)} = \mathcal{L} := K \wedge I$  and for  $k \geq 1$ ,  $\mathcal{L}^{(k)} = \widehat{\mathcal{L}^{(k-1)}}$ . Let  $\varepsilon(K)$  be equal to 1 if  $K \subset \mathcal{L}_+$  and  $-1$  if  $K \subset \mathcal{L}_-$ . Then, we have*

$$(i) \quad h_{\mathcal{L}}(I) \text{III} h_{\mathcal{L}}(K) = \begin{cases} \frac{-1}{|\mathcal{L}|} & \text{if } K \subset (\mathcal{L}_+)_+ \\ \frac{1}{|\mathcal{L}|} & \text{if } K \subset (\mathcal{L}_+)_- \\ \frac{-1}{|\mathcal{L}|} & \text{if } K \subset (\mathcal{L}_-)_+ \\ \frac{1}{|\mathcal{L}|} & \text{if } K \subset (\mathcal{L}_-)_-; \end{cases}$$

- (ii)  $h_{\mathcal{L}^{(1)}}(I) \text{III} h_{\mathcal{L}^{(1)}}(K) = \frac{\varepsilon(K)}{2|\mathcal{L}|}$ ;
- (iii) and for  $k \geq 2$ ,  $h_{\mathcal{L}^{(k)}}(I) \text{III} h_{\mathcal{L}^{(k)}}(K) = \begin{cases} \frac{1}{2^k|\mathcal{L}|} & \text{if } \mathcal{L}^{(k-2)} = \mathcal{L}_+^{(k-1)} \\ -\frac{1}{2^k|\mathcal{L}|} & \text{if } \mathcal{L}^{(k-2)} = \mathcal{L}_-^{(k-1)}. \end{cases}$

*Proof.* It is enough to deal with the case  $K \subset \mathcal{L}_+$  (i.e.  $\varepsilon(K) = 1$ ). Since  $I \cap K = \emptyset$  and by the definition of  $\mathcal{L}$ , we have  $I \subset \mathcal{L}_-$  and  $h_{\mathcal{L}}(I) = \frac{-1}{\sqrt{|\mathcal{L}|}}$ . Now, there are only two cases to consider for  $K$ : either  $K \subset (\mathcal{L}_+)_+$  and  $\text{III} h_{\mathcal{L}}(K) = \frac{1}{\sqrt{|\mathcal{L}|}}$  or  $K \subset (\mathcal{L}_+)_-$  and  $\text{III} h_{\mathcal{L}}(K) = \frac{-1}{\sqrt{|\mathcal{L}|}}$ . It follows that

$$h_{\mathcal{L}}(I) \text{III} h_{\mathcal{L}}(K) = \begin{cases} -\frac{1}{|\mathcal{L}|} & \text{if } K \subset (\mathcal{L}_+)_+ \\ \frac{1}{|\mathcal{L}|} & \text{if } K \subset (\mathcal{L}_+)_-. \end{cases}$$

Suppose first that  $\mathcal{L} = \mathcal{L}_+^{(1)}$ . Then, we have  $I \subset \mathcal{L} = \mathcal{L}_+^{(1)}$  and  $K \subset \mathcal{L}_+ = (\mathcal{L}_+^{(1)})_+$  which implies that  $h_{\mathcal{L}^{(1)}}(I) \text{III} h_{\mathcal{L}^{(1)}}(K) = \frac{1}{|\mathcal{L}^{(1)}|}$  with  $h_{\mathcal{L}^{(1)}}(I) = \text{III} h_{\mathcal{L}^{(1)}}(K) = \frac{1}{\sqrt{|\mathcal{L}^{(1)}|}}$ . On the other hand, if  $\mathcal{L} = \mathcal{L}_-^{(1)}$  then we have  $I \subset \mathcal{L} = \mathcal{L}_-^{(1)}$  and  $K \subset \mathcal{L}_+ = (\mathcal{L}_-^{(1)})_+$ . We still obtain that  $h_{\mathcal{L}^{(1)}}(I) \text{III} h_{\mathcal{L}^{(1)}}(K) = \frac{1}{|\mathcal{L}^{(1)}|}$  with  $h_{\mathcal{L}^{(1)}}(I) = \text{III} h_{\mathcal{L}^{(1)}}(K) = \frac{-1}{\sqrt{|\mathcal{L}^{(1)}|}}$ .

Let us prove property (iii) for  $k \geq 2$ . Suppose first that  $\mathcal{L}^{(k-2)} = \mathcal{L}_+^{(k-1)}$ . When  $\mathcal{L}^{(k-1)} = \mathcal{L}_+^{(k)}$ , we have that  $I \subset \mathcal{L}^{(k-1)} = \mathcal{L}_+^{(k)}$  and  $K \subset \mathcal{L}_+^{(k-1)} = (\mathcal{L}_+^{(k)})_+$  which implies that  $h_{\mathcal{L}^{(k)}}(I) \text{III} h_{\mathcal{L}^{(k)}}(K) = \frac{1}{|\mathcal{L}^{(k)}|}$  with  $h_{\mathcal{L}^{(k)}}(I) = \text{III} h_{\mathcal{L}^{(k)}}(K) = \frac{1}{\sqrt{|\mathcal{L}^{(k)}|}}$ . And, when  $\mathcal{L}^{(k-1)} = \mathcal{L}_-^{(k)}$ , we have that  $I \subset \mathcal{L}^{(k-1)} = \mathcal{L}_-^{(k)}$  and  $K \subset \mathcal{L}_+^{(k-1)} = (\mathcal{L}_-^{(k)})_+$  which implies that  $h_{\mathcal{L}^{(k)}}(I) \text{III} h_{\mathcal{L}^{(k)}}(K) = \frac{1}{|\mathcal{L}^{(k)}|}$  with  $h_{\mathcal{L}^{(k)}}(I) = \text{III} h_{\mathcal{L}^{(k)}}(K) = \frac{-1}{\sqrt{|\mathcal{L}^{(k)}|}}$ . One can easily deduce the case  $\mathcal{L}^{(k-2)} = \mathcal{L}_-^{(k-1)}$  which leads to  $h_{\mathcal{L}^{(k)}}(I) \text{III} h_{\mathcal{L}^{(k)}}(K) = \frac{-1}{|\mathcal{L}^{(k)}|}$ .  $\square$

Let us now prove the first sub-case.

**Lemma 4.2.** *We suppose that  $K, I \subset \mathbb{R}_+$ ,  $K \cap I = \emptyset$ . Let  $\mathcal{L} = K \wedge I$  and assume that  $\mathcal{L} = [0, 2^N)$  for some  $N \in \mathbb{Z}$ .*

(1) *Assume that  $I \subset \mathcal{L}_+$  while  $K \subsetneq \mathcal{L}_-$ . Then*

(a) *If  $K \subset \mathcal{L}_{--}$  then  $\mathbf{1}_K \text{III} \mathbf{1}_I = 0$  thus  $\frac{\|\mathbf{1}_K \text{III} \mathbf{1}_I\|^2}{|I|} = 0$ ;*

(b) *If  $K \subset \mathcal{L}_{-+}$  then  $\mathbf{1}_K \text{III} \mathbf{1}_I = -\frac{2|I|}{|\mathcal{L}|} \mathbf{1}_K$  thus  $\frac{\|\mathbf{1}_K \text{III} \mathbf{1}_I\|^2}{|I|} = 4 \frac{|I||K|}{|\mathcal{L}|^2}$ .*

(2) *Assume that  $I \subset \mathcal{L}_-$  while  $K \subset \mathcal{L}_{+\pm}$ . Then  $\mathbf{1}_K \text{III} \mathbf{1}_I = \pm \frac{|I|}{|\mathcal{L}|} \mathbf{1}_K$  thus  $\frac{\|\mathbf{1}_K \text{III} \mathbf{1}_I\|^2}{|I|} = \frac{|I||K|}{|\mathcal{L}|^2}$ .*

*Proof.* Now let again  $\mathcal{L}^{(k)}$  be defined by  $\mathcal{L}^{(0)} = \mathcal{L}$  and  $\mathcal{L}^{(k+1)} = \widehat{\mathcal{L}^{(k)}}$ . Note that, as  $\mathcal{L} = [0, 2^{N^0})$ ,  $\mathcal{L}^{(k)} = \mathcal{L}_-^{(k+1)}$ . As  $\widehat{K} = \mathcal{L}_\pm \neq \mathcal{L}$ , we want to estimate

$$\frac{1}{|I|} \mathbf{1}_K \text{III} \mathbf{1}_I = \left( \sum_{L \supset \mathcal{L}} h_L(I) \text{III} h_L(K) \right) \mathbf{1}_K = \left( \sum_{k \geq 0} h_{\mathcal{L}^{(k)}}(I) \text{III} h_{\mathcal{L}^{(k)}}(K) \right) \mathbf{1}_K.$$

Assume first that  $K \subset \mathcal{L}_{--}$  and  $I \subset \mathcal{L}_+$ . Then, according to the previous lemma,

$$h_{\mathcal{L}^{(0)}}(I) \text{III} h_{\mathcal{L}^{(0)}}(K) = \frac{1}{|\mathcal{L}|} \quad \text{while} \quad h_{\mathcal{L}^{(k)}}(I) \text{III} h_{\mathcal{L}^{(k)}}(K) = \frac{-1}{2^k |\mathcal{L}|}$$

for  $k \geq 1$ . The result follows immediately.

Assume now that  $K \subset (\mathcal{L}_-)_+$  and  $I \subset \mathcal{L}_+$ . Then, according to the previous lemma again,

$$h_{\mathcal{L}^{(k)}}(I) \text{III} h_{\mathcal{L}^{(k)}}(K) = \frac{-1}{2^k |\mathcal{L}|}$$

for  $k \geq 0$ . The result again follows immediately.

Let us now assume that  $K \subset (\mathcal{L}_+)_\pm$  and  $I \subset \mathcal{L}_-$ . Then, according to the previous lemma,

$$h_{\mathcal{L}^{(0)}}(I) \text{III} h_{\mathcal{L}^{(0)}}(K) = \frac{\pm 1}{|\mathcal{L}|}$$

while

$$h_{\mathcal{L}^{(1)}}(I) \text{III} h_{\mathcal{L}^{(1)}}(K) = \frac{1}{2|\mathcal{L}|} \quad \text{and} \quad h_{\mathcal{L}^{(k)}}(I) \text{III} h_{\mathcal{L}^{(k)}}(K) = \frac{-1}{2^k |\mathcal{L}|} \quad k \geq 2$$

and the result again follows immediately.  $\square$

Now if  $I \subset \mathcal{D}$ , there exists  $M_0$  such that  $I \subset [0, 2^{M_0}]$  but  $I \not\subset [0, 2^{M_0-1}]$ . In the case  $I = [0, 2^{M_0}]$ , the previous lemma determines  $\text{III} \mathbf{1}_I$  on  $I^c$ . Otherwise  $I \subset [2^{M_0-1}, 2^{M_0}]$  and the previous lemma determines  $H \mathbf{1}_I$  on  $[0, 2^{M_0-1}]$  and on  $[2^{M_0}, +\infty)$ .

It remains to consider the case  $K, I$  such that  $K \cap I = \emptyset$  and  $K, I \subset [2^{M_0-1}, 2^{M_0}]$ . We keep the same notation:  $\mathcal{L} = K \wedge I$  for the first common ancestor of  $K$  and  $I$ ,  $\mathcal{L}^{(0)} = \mathcal{L}$  and  $\mathcal{L}^{(k)} = \widehat{\mathcal{L}^{(k-1)}}$  for  $k \geq 1$ . We further write  $\mathcal{L}^* = [0, 2^{M_0}]$  the first common ancestor of  $K, I$  of the form  $[0, 2^M]$  so that  $K \wedge I \subset \mathcal{L}_+^*$ . Let  $k^*$  be defined by  $\mathcal{L}^* = \mathcal{L}^{(k^*)}$ . It follows that  $2^{M_0} = |\mathcal{L}^*| = 2^{k^*} |\mathcal{L}| = 2^{k^*} |K \wedge I|$ . Now

$$\begin{aligned} \frac{1}{|I|} \mathbf{1}_K \text{III} \mathbf{1}_I &= \mathbf{1}_K \sum_{L \supset \mathcal{L}} h_L(I) \text{III} h_L(K) \\ &= \mathbf{1}_K \left( \sum_{L \supseteq \mathcal{L}^*} h_L(I) \text{III} h_L(K) + \sum_{\mathcal{L} \subset L \subset \mathcal{L}^*} h_L(I) \text{III} h_L(K) \right) \\ &= \mathbf{1}_K \sum_{\mathcal{L} \subset L \subset \mathcal{L}^*} h_L(I) \text{III} h_L(K). \end{aligned}$$

Indeed, if  $L = \widehat{\mathcal{L}^*} = \mathcal{L}^{(k^*+1)}$  then  $\mathcal{L}^{(k^*-1)} \subset \mathcal{L}_+^{(k^*)}$  so that, according to Lemma 4.1,

$$h_L(I) \text{III} h_L(K) = \frac{1}{2^{k^*+1} |\mathcal{L}|}.$$

On the other hand, if  $L = \mathcal{L}^{(k)}$  for  $k \geq k^* + 2$ ,  $\mathcal{L}^{(k-2)} \subset \mathcal{L}_-^{(k-1)}$  so that

$$h_L(I) \text{III} h_L(K) = -\frac{1}{2^{k^*+1} |\mathcal{L}|}.$$

Therefore,  $\sum_{L \supseteq \mathcal{L}^*} h_L(I) \text{III} h_L(K) = 0$ .

We now distinguish 2 cases. First assume that  $\mathcal{L} = \mathcal{L}_+^*$ . Then

$$\frac{1}{|I|} \mathbf{1}_K \mathbb{I} \mathbf{1}_I = \mathbf{1}_K (h_{\mathcal{L}_+^*}(I) \mathbb{I} h_{\mathcal{L}_+^*}(K) + h_{\mathcal{L}^*}(I) \mathbb{I} h_{\mathcal{L}^*}(K))$$

Applying Lemma 4.1 we get

$$\frac{1}{|I|} \mathbf{1}_K \mathbb{I} \mathbf{1}_I = \begin{cases} -\frac{1}{2|\mathcal{L}|} \mathbf{1}_K & \text{if } I \subset \mathcal{L}_-, K \subset (\mathcal{L}_+)_+ \\ \frac{3}{2|\mathcal{L}|} \mathbf{1}_K & \text{if } I \subset \mathcal{L}_-, K \subset (\mathcal{L}_+)_- \\ -\frac{3}{2|\mathcal{L}|} \mathbf{1}_K & \text{if } I \subset \mathcal{L}_+, K \subset (\mathcal{L}_-)_+ \\ \frac{1}{2|\mathcal{L}|} \mathbf{1}_K & \text{if } I \subset \mathcal{L}_-, K \subset (\mathcal{L}_-)_-. \end{cases}$$

Let us now assume that  $\mathcal{L} \subsetneq \mathcal{L}_+^*$ . Then each  $L$  with  $\mathcal{L} \subset L \subset \mathcal{L}^*$  is of the form  $L = \mathcal{L}^{(k)}$  with  $0 \leq k \leq k^*$  and for each such  $k$ , there is an  $\varepsilon_k = \pm 1$  such that  $h_L(I) \mathbb{I} h_L(K) = \frac{\varepsilon_k}{2^k |\mathcal{L}|}$ . But then

$$\begin{aligned} \left| \frac{1}{|I|} \mathbf{1}_K \mathbb{I} \mathbf{1}_I \right| &= \mathbf{1}_K \left| \sum_{k=0}^{k^*} \frac{\varepsilon_k}{2^k |\mathcal{L}|} \right| = \frac{\mathbf{1}_K}{|\mathcal{L}|} \left( 1 + \sum_{k=1}^{k^*} \frac{\varepsilon_0 \varepsilon_k}{2^k} \right) \\ &\geq \frac{\mathbf{1}_K}{|\mathcal{L}|} \left( 1 - \sum_{k=1}^{k^*} 2^{-k} \right) = \frac{\mathbf{1}_K}{|\mathcal{L}|} \frac{|K \wedge I|}{2^{M_0}} \end{aligned}$$

so that

$$\frac{\|\mathbf{1}_K \mathbb{I} \mathbf{1}_I\|_{L^2}^2}{|I|} \geq \left( \frac{|K \wedge I|}{2^{M_0}} \right)^2 \frac{|I||K|}{|\mathcal{L}|^2}.$$

We can now summarize the results of this section:

**Theorem 4.3.** *Let  $\eta > 0$ . Let  $I, K \in \mathcal{D}$  be such that  $I \subset \mathbb{R}^+$  and let  $M_0$  be the smallest integer such that  $I \subset [0, 2^{M_0}]$ . Let  $f_0 \in W^{1,2}(I)$  be such that  $|I| \|f_0'\|_{L^2(I)} \leq 2\pi\eta \|f_0\|_{L^2(I)}$  and let  $f$  be the extension of  $f_0$  by 0. Then*

- (i) *If  $K \subset \mathbb{R}_-$  then  $\mathbf{1}_K \mathbb{I} f = 0$ .*
- (ii) *If  $K \subset [2^{M_0+k}, 2^{M_0+k+1}]$  then*

$$\|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 \geq (1 - \eta)^2 \frac{|I||K|}{2^{2(M_0+k)}} \|f\|_{L^2}^2.$$

- (iii) *If  $I \subset [2^{M_0-1}, 2^{M_0}]$  then*
  - (a) *If  $K \subset [0, 2^{M_0-2}]$  then  $\mathbf{1}_K \mathbb{I} f = 0$ ;*
  - (b) *If  $K \subset [2^{M_0-2}, 2^{M_0-1}]$  then*

$$\|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 \geq (1 - \eta)^2 \frac{|I||K|}{2^{2(M_0-1)}} \|f\|_{L^2}^2;$$

- (c)  *$K \subset [2^{M_0-1}, 2^{M_0}]$  and  $K \cap I = \emptyset$  then*

$$\|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 \geq (1 - \eta)^2 \frac{|I||K||K \wedge I|^2}{2^{4M_0}} \|f\|_{L^2}^2.$$

*In all of the above cases, no estimate of the form  $\|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 \geq C \|f\|_{L^2}^2$  can hold for all functions  $f \in L^2$  with support in  $I$ .*

5. THIRD CASE:  $K \subsetneq I$ 

For  $K \subsetneq I$ , we write  $\varepsilon(K, I) = +1$  if  $K \subset I_+$  and  $\varepsilon(K, I) = -1$  if  $K \subset I_-$ . According to Lemma 2.1,  $\mathbf{1}_K \mathbb{I} \mathbf{1}_I = \varepsilon(K, I) \mathbf{1}_K$ , in particular,  $\|\mathbf{1}_K \mathbb{I} \mathbf{1}_I\|_{L^2}^2 = |K|$ .

From equation (2.6) and Lemma 2.1 we get that

$$\begin{aligned}
 \mathbf{1}_K \mathbb{I} f &= \langle f \rangle_I \mathbf{1}_K \mathbb{I} \mathbf{1}_I + \sum_{K \subsetneq J \subset I} \widehat{f}(J) \mathbf{1}_K \mathbb{I} h_J + \sum_{J \subset K} \widehat{f}(J) \mathbf{1}_K \mathbb{I} h_J \\
 (5.1) \quad &= \left[ \langle f \rangle_I \varepsilon(K, I) + \sum_{\widehat{K} \subsetneq J \subset I} \widehat{f}(J) \mathbb{I} h_J(K) \right] \mathbf{1}_K + \frac{\varepsilon(K)}{\sqrt{2}} \widehat{f}(\widehat{K}) h_K + \sum_{J \subset K} \widehat{f}(J) \mathbb{I} h_J.
 \end{aligned}$$

Let us denote by  $\mathcal{B}$  the subspace  $\overline{\text{span}} \{ \mathbb{I} h_J, J \subset K \}$  and  $P_{\mathcal{B}}$  the orthogonal projection onto  $\mathcal{B}$ . Observe that for  $J \subset K$ ,  $\langle \mathbf{1}_K \mathbb{I} f, \mathbb{I} h_J \rangle = \langle \mathbb{I} f, \mathbb{I} h_J \rangle = \widehat{f}(J)$ . Therefore

$$(5.2) \quad P_{\mathcal{B}}(\mathbf{1}_K \mathbb{I} f) = \sum_{J \subset K} \langle \mathbf{1}_K \mathbb{I} f, \mathbb{I} h_J \rangle \mathbb{I} h_J = \sum_{J \subset K} \widehat{f}(J) \mathbb{I} h_J.$$

Moreover, the  $h_J$ 's being orthonormal and  $\mathbb{I}$  being unitary,

$$(5.3) \quad \|P_{\mathcal{B}}(\mathbf{1}_K \mathbb{I} f)\|_{L^2}^2 = \sum_{J \subset K} |\widehat{f}(J)|^2.$$

On the other hand, from (5.1) and (5.2), it follows that

$$(I - P_{\mathcal{B}})(\mathbf{1}_K \mathbb{I} f) = \left[ \langle f \rangle_I \varepsilon(K, I) + \sum_{\widehat{K} \subsetneq J \subset I} \widehat{f}(J) \mathbb{I} h_J(K) \right] \mathbf{1}_K + \frac{\varepsilon(K)}{\sqrt{2}} \widehat{f}(\widehat{K}) h_K.$$

But  $h_K$  and  $\mathbf{1}_K$  are orthogonal so that

$$(5.4) \quad \|(I - P_{\mathcal{B}})(\mathbf{1}_K \mathbb{I} f)\|_{L^2}^2 = \left[ \langle f \rangle_I \varepsilon(K, I) + \sum_{\widehat{K} \subsetneq J \subset I} \widehat{f}(J) \frac{\varepsilon(K, J)}{\sqrt{|J|}} \right]^2 |K| + \frac{|\widehat{f}(\widehat{K})|^2}{2}.$$

We can now prove the following:

**Theorem 5.1.** *Let  $I, K \in \mathcal{D}$  be such that  $K \subset I$ . Then, for every  $f \in L^2(\mathbb{R})$  with  $\text{supp } f \subset I$ ,*

$$(5.5) \quad \|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 = \left[ \langle f \rangle_I \varepsilon(K, I) + \sum_{\widehat{K} \subsetneq J \subset I} \widehat{f}(J) \frac{\varepsilon(K, J)}{\sqrt{|J|}} \right]^2 |K| + \frac{|\widehat{f}(\widehat{K})|^2}{2} + \sum_{J \subset K} |\widehat{f}(J)|^2.$$

In particular,

- (i) for every  $f \in L^2(\mathbb{R})$ ,  $\|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 \geq \|\mathbf{1}_K f\|_{L^2}^2$  and  $\|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 \geq \frac{1}{2} \|\mathbf{1}_{\widehat{K}} f\|_{L^2}^2$ .
- (ii) If  $I \supsetneq \widehat{K}$ , there exists no constant  $C = C(K, I)$  such that, for every  $f \in L^2(\mathbb{R})$  with  $\text{supp } f \subset I$ ,  $\|\mathbf{1}_K \mathbb{I} f\|_{L^2} \geq C \|f\|_{L^2}$ .

*Proof.* As  $\|\mathbf{1}_K \mathbb{I} f\|_{L^2}^2 = \|P_{\mathcal{B}}(\mathbf{1}_K \mathbb{I} f)\|_{L^2}^2 + \|(I - P_{\mathcal{B}})(\mathbf{1}_K \mathbb{I} f)\|_{L^2}^2$ , (5.5) is a direct combination of (5.3) and (5.4). The inequalities (i) are direct consequences of (5.5).

For the last part of the proposition, let  $f = -\frac{\varepsilon(K, I)}{\sqrt{|I|}} \mathbf{1}_I + h_I$ . Then  $f \in L^2(\mathbb{R})$  is supported in  $I$  and  $\widehat{f}(J) = \delta_{I, J}$  if  $J \subset I$  and  $\langle f \rangle_I = -\frac{\varepsilon(K, I)}{\sqrt{|I|}}$ . Further (5.5) shows that  $\|\mathbf{1}_K \mathbb{H} f\|_{L^2} = 0$  while  $\|f\|_{L^2} = \sqrt{2}$ .  $\square$

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